

## A GEOMETRIC INTERPRETATION OF ONE-DIMENSIONAL QUASINORMAL RINGS

Luciana RAMELLA

*Istituto Matematico Università, via L.B. Alberti 4, Genova, Italy*

Communicated by H. Bass

Received 28 July 1983

### Introduction

A commutative noetherian reduced ring  $A$  is said to be *seminormal* (SN) [resp. *quasinormal* (QN)] if its integral closure is a finite  $A$ -module and the canonical homomorphism of the Picard groups  $\text{Pic } A \rightarrow \text{Pic } A[T]$  [resp.  $\text{Pic } A \rightarrow \text{Pic } A[T, T^{-1}]$ ] is an isomorphism.

In what follows,  $A$  is a reduced, 1-dimensional noetherian ring with integral closure  $\bar{A}$ , a finite  $A$ -module.

We define  $v(A) = h_0(\bar{A}/c) - h_0(A/c) - h_0(\bar{A}) + h_0(A)$ , where  $c$  is the conductor of  $A$  in  $\bar{A}$  and  $h_0(A)$  denotes the number of the connected components of  $\text{Spec}(A)$ .

Bass and Murty proved in [2] the following

**Theorem 1.** (a)  $\text{Pic } A[T] \cong \text{Pic } A \oplus X$ ;  $X = 0$  if and only if  $\sqrt[A]{c} = c$ .

(b)  $\text{Pic } A[T, T^{-1}] \cong \text{Pic } A \oplus X \oplus Y$  and  $Y$  is a free abelian group whose rank is  $v(A)$ .

As a consequence of Theorem 1 we have:

**Theorem 2.**  $A$  is QN if and only if  $A$  is SN and  $v(A) = 0$ .

In Section 1, by using combinatorial methods, we give a geometric interpretation of  $v(A)$ . This allows a geometric description of a quasinormal ring of dimension 1. Thus we get an affirmative answer to a problem posed by Greco in [5].

In Section 2, we study relations between the numbers  $v(A)$  and  $v(B)$ , where  $B$  is a finite étale  $A$ -algebra. We show that quasinormality is not so well behaved as seminormality.

Let  $p_1, \dots, p_n$  be the minimal primes of  $A$ . Let  $A_i = A/p_i$  and assume  $\dim A_i = 1$  for every  $i = 1, \dots, n$ . The affine curves  $X_i = \text{Spec}(A_i)$ , for  $i = 1, \dots, n$ , are the irreducible components of the curve  $X = \text{Spec}(A)$ .

$\bar{X} = \text{Spec}(\bar{A})$  is the normalization of  $X$ ; we have  $\bar{X} = \prod_{i=1}^n \bar{X}_i$ , since  $\bar{A} = \bigoplus_{i=1}^n \bar{A}_i$ .

Let  $P_1, \dots, P_m$  be the singular points of  $X$  and let  $x_1, \dots, x_M$  be the branches of  $X$  ( $x \in \bar{X}$  is a branch of  $X$  over  $P$  if  $x \in \phi^{-1}(P)$ , where  $\phi: \bar{X} \rightarrow X$  is the canonical morphism and  $P$  is a singular point of  $X$ ).

We have  $n = h_0(\bar{A})$ ,  $m = h_0(A/c)$  and  $M = h_0(\bar{A}/c)$ .

### 1. Incidence cycles of a curve

In this section, we give a geometric interpretation of the number  $v(A)$ , by using combinatorial methods concerning graph theory. Similar methods are also applied by Roberts in [11] and [12] to calculate the Picard group of a class of curves.

We associated to the curve  $X = \text{Spec}(A)$  the matrices  $\beta = (\beta_{ij}) \in \mathbb{Z}_2^{M \times m}$  and  $\gamma = (\gamma_{ij}) \in \mathbb{Z}_2^{M \times n}$ , where  $\beta_{ij}$  is 1 if  $\phi(x_i) = P_j$  and 0 otherwise,  $\gamma_{ij}$  is 1 if  $x_i \in \bar{X}_j$  and 0 otherwise.

We say that  $\alpha = (\beta, \gamma)$  is the *matrix associated to  $X$*  (or to  $A$ ).

**Proposition 1.1.** *Let  $\alpha$  be the matrix associated to the curve  $X$ , let  $\rho(\alpha)$  be the rank of the matrix  $\alpha$ , we have  $\rho(\alpha) < m + n$ . If the curve  $X$  is connected, then  $\rho(\alpha) = m + n - 1$ .*

**Proof.** Every row of the matrix  $\alpha = (\beta, \gamma)$  contains two 1's, one in  $\beta$  and one in  $\gamma$ . Then the sum of columns of  $\beta$  equals the sum of the columns of  $\gamma$  and we have  $\rho(\alpha) < m + n$ . If  $X$  is connected it is easily shown that the above column relation is the only one, up to scalar factors.

We associated to the curve  $X = \text{Spec}(A)$  the graph  $\Gamma$  whose vertices are  $P_1, \dots, P_m, X_1, \dots, X_n$  and whose edges represent the  $M$  branches of  $X$  in this way: if  $x_r$  is a branch over  $P_i$  and  $x_r \in \bar{X}_j$  we construct an edge by joining  $P_i$  and  $X_j$ .

Observe that several edges may join two vertices.

The graph  $\Gamma$  just constructed is said to be the *graph associated to  $X$*  (or to  $A$ ).

Let  $\alpha$  be the matrix and let  $\Gamma$  be the graph associated to  $X$ , then  $\alpha$  is the incidence matrix of the graph  $\Gamma$ .

It is known that (see e.g. [3]):

(a) The number  $v(\Gamma)$  of independent cycles of the graph  $\Gamma$  is equal to the number of the edges of  $\Gamma$  minus the rank of the matrix  $\alpha$ .

(b) The rank of  $\alpha$  is equal to the number of the edges of  $\Gamma$  minus the number of the connected components of  $\Gamma$ .

One can prove the following

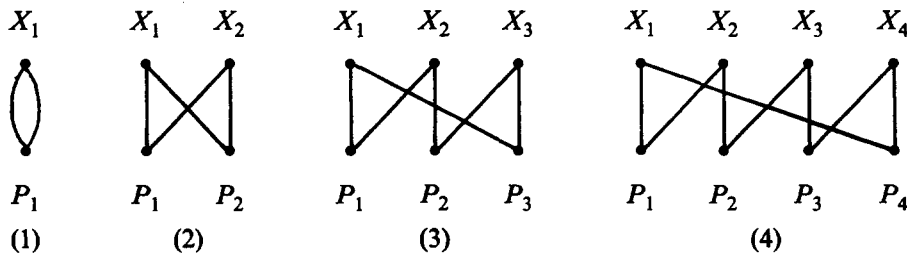
**Proposition 1.2.** *Let  $\Gamma$  be the graph associated to the curve  $X = \text{Spec}(A)$ . If  $X$  is connected, then also the graph  $\Gamma$  is connected and we have  $v(\Gamma) = M - m - n + 1$ .*

Then we can conclude that  $v(A)$  coincides with the number of independent cycles of the graph  $\Gamma$  associated to the curve  $X = \text{Spec}(A)$ .

**Theorem 1.3.** *Let  $\Gamma$  be the graph associated to the curve  $X$ .  $\Gamma$  contains cycles if and only if  $X$  satisfies one of the following conditions:*

- (a) *An irreducible component of  $X$  is not locally unibranche.*
- (b) *Two irreducible components of  $X$  meet in more then one point.*
- (c)  *$X$  contains polygons.*

**Proof.** An ‘elementary’ cycle of the graph  $\Gamma$  is a cycle of one of the following types



The cycle  $X_1 P_1 X_1$  of the case (1) represents two distinct branches belonging to  $\bar{X}_1$  and lying over the same singular point  $P_1$ , therefore  $X_1$  is not locally unibranche.

The cycle  $X_1 P_1 X_2 P_2 X_1$  of the case (2) represents two distinct points belonging to the intersection of two irreducible components of  $X$ .

If  $r \geq 3$ , the cycle  $X_1 P_1 \cdots X_r P_r X_1$  represents a polygon of  $X$  with  $r$  vertices.

The following corollary gives an affirmative answer to a problem posed by Greco in [5, 4.12].

**Corollary 1.4.**  *$A$  is QN if and only if  $A$  is SN and the following conditions are satisfied:*

- (i) *All irreducible components of  $X = \text{Spec}(A)$  are locally unibranche.*
- (ii) *Two distinct irreducible components of  $X$  meet in at most one point.*
- (iii)  *$X$  does not contain polygons.*

**Proof.**  $A$  is QN if and only if  $A$  is SN and  $v(A) = 0$ . Moreover  $v(A) = v(\Gamma)$ , where  $\Gamma$  is the graph associated to  $A$ , then the conclusion follows from Theorem 1.3.

**Definition 1.5.** Any cycle of the graph  $\Gamma$  associated to  $X = \text{Spec}(A)$  is said to be an *incidence cycle* of  $X$  (or of  $A$ )

Then an incidence cycle of  $X$  is a composition of cycles of the types (1), (2), (3), ... described above. Thus  $v(A)$  represents the number of the independent incidence cycles of  $A$ .

We can restate Theorem 2 in the following way: “ $A$  is QN if and only if  $A$  is SN and  $A$  has no incidence cycle”.

If  $A$  is an integral domain,  $A$  has no incidence cycle if and only if  $A$  is locally unibranche. Therefore we also get the following: “If  $A$  is an integral domain,  $A$  is QN if and only if  $A$  is SN and locally unibranche” (see [5]).

Now we recall the following results about seminormality (see e.g. [6]):

- (i) If  $A$  is SN, then  $A_S$  is SN for every multiplicative subset  $S$  of  $A$ .
- (ii)  $A$  is SN if and only if  $A_{\mathfrak{m}}$  is SN for every maximal ideal  $\mathfrak{m}$  of  $A$ .
- (iii) If  $A$  is SN, any intermediate ring  $A \subset B \subset \bar{A}$  is SN.

We will see that (i) and (iii) also hold for quasinormality, but (ii) may not.

**Proposition 1.6.** (a) *Let  $S$  be a multiplicative subset of  $A$ , then we have  $v(A_S) \leq v(A)$ .*

(b)  *$A_i$  is locally unibranche for every  $i = 1, \dots, n$  if and only if  $v(A_{\mathfrak{m}}) = 0$  for every maximal ideal  $\mathfrak{m}$  of  $A$ .*

**Proof.** (a) Since  $\overline{(A_S)} = \bar{A}_S$  and  $cA_S$  is the conductor of  $A_S$  in  $\bar{A}_S$ , where  $c$  is the conductor of  $A$  in  $\bar{A}$ , the graph  $\Gamma_{A_S}$  associated to  $A_S$  is a subgraph of the graph  $\Gamma_A$  associated to  $A$ .

(b) There is maximal ideal  $\mathfrak{m}$  of  $A$  such that  $v(A_{\mathfrak{m}}) \neq 0$  if and only if there is an irreducible component  $X_i = \text{Spec}(A_i)$  of  $X = \text{Spec}(A)$  with two distinct branches lying over a singular point  $P$  of  $X_i$ .

**Corollary 1.7.** (a) *Let  $S$  be a multiplicative subset of  $A$ . If  $A$  is QN, then  $A_S$  is QN.*

(b)  *$A$  is SN and  $A_i$  is locally unibranche for every  $i = 1, \dots, n$  if and only if  $A_{\mathfrak{m}}$  is QN for every maximal ideal  $\mathfrak{m}$  of  $A$ .*

Notice that the ring  $A = \mathbb{C}[X, Y]/(Y-1)(Y-X^2)$  is not QN, but  $A_{\mathfrak{m}}$  is QN for every maximal ideal  $\mathfrak{m}$  of  $A$ .

If  $A$  is an integral domain, then  $A$  is QN if and only if  $A_{\mathfrak{m}}$  is QN for every maximal ideal  $\mathfrak{m}$  of  $A$  (see [5]).

**Proposition 1.8.** *Let  $B$  be a subring of  $\bar{A}$  containing  $A$ , then we have  $v(B) \leq v(A)$ .*

**Proof.** If the maximal ideals  $\mathfrak{M}_1, \dots, \mathfrak{M}_r$  of  $B$ , the minimal prime ideals  $\mathfrak{P}_1, \dots, \mathfrak{P}_r$  of  $B$  and the maximal ideals  $M_1, \dots, M_{2r}$  of  $\bar{B} = \bar{A}$  form an elementary cycle of the graph  $\Gamma_B$  associated to  $B$ , then  $\mathfrak{M}_1 \cap A, \dots, \mathfrak{M}_r \cap A, \mathfrak{P}_1 \cap A, \dots, \mathfrak{P}_r \cap A, M_1, \dots, M_{2r}$  form a cycle of the graph  $\Gamma_A$  associated to  $A$ .

**Corollary 1.9.** (a) *If  $A$  is QN, then any intermediate ring  $A \subset B \subset \bar{A}$  is QN.*

(b) *If  $v(A) = 0$ , then the seminormalization of  $A$  is QN.*

## 2. Etale coverings and incidence cycles

We recall the following results (see [6]).

- (a) Let  $f: A \rightarrow B$  be a faithfully flat homomorphism. If  $B$  is SN, then  $A$  is SN.
- (b) Let  $f: A \rightarrow B$  be an étale homomorphism. If  $A$  is SN, then  $B$  is SN.

Let  $A$  and  $B$  be (1-dimensional) integral domains and let  $f: A \rightarrow B$  be a faithfully flat homomorphism. If  $B$  is QN, then  $A$  is QN (see [5, 4.10]).

If  $B$  is not an integral domain, the above assertion is false. Indeed let  $A = \mathbb{C}[X, Y]/(Y^2 - X^2(X - 1))$  be the coordinate ring of the cubic node in  $\mathbb{A}_{\mathbb{C}}^2$  and let  $\mathfrak{m}$  be the maximal ideal of  $A$  corresponding to the singular point 0. Let  $\hat{A}_{\mathfrak{m}}$  be the completion of the local ring  $A_{\mathfrak{m}}$ . Then the canonical homomorphism  $f: A_{\mathfrak{m}} \rightarrow \hat{A}_{\mathfrak{m}}$  is a faithfully flat homomorphism,  $\hat{A}_{\mathfrak{m}}$  is QN,  $A_{\mathfrak{m}}$  is SN but it is not locally unibranch, hence  $A_{\mathfrak{m}}$  is not QN.

Quasinormality is not preserved by étale extensions. The ring  $A = \mathbb{R}[X, Y]_{(X, Y)}/(X^2 + Y^2)$  is QN, but there exists a local finite étale  $A$ -algebra  $B$  which is SN but not QN (see [5, 4.13]).

From now on the ring  $A$  will be assumed to be a finitely generated  $k$ -algebra, where  $k$  is an algebraically closed field.

In this section we study the incidence cycles of a finite étale  $A$ -algebra  $B$ , and therefore the incidence cycles of an étale covering of  $X = \text{Spec}(A)$ .

We simply say that a homomorphism  $f: A \rightarrow B$  is an étale covering (of  $A$ ) if the induced morphism  $F: \text{Spec}(B) \rightarrow \text{Spec}(A)$  is an étale covering (of  $\text{Spec}(A)$ ).

**Proposition 2.1.** *Let  $f: A \rightarrow B$  be an étale covering. Then we have  $v(B) \geq v(A)$ .*

**Proof.** We may assume that  $A$  and  $B$  are connected. Let  $q$  be the degree of  $f$ . Since the ground field  $k$  is algebraically closed, any fiber of the morphism  $F: \text{Spec}(B) \rightarrow \text{Spec}(A)$  at a closed point consists of  $q$  points. Since  $f$  is an étale homomorphism, we have  $c_A B = c_B$  and  $\bar{B} = B \otimes_A \bar{A}$ , where  $c_A$  and  $c_B$  denote the conductor of  $A$  in  $\bar{A}$  and of  $B$  in  $\bar{B}$  respectively. The induced homomorphisms  $A/c_A \rightarrow B/c_B$ ,  $\bar{A} \rightarrow \bar{B}$ ,  $\bar{A}/c_A \rightarrow \bar{B}/c_B$  are étale coverings of degree  $q$ . Then  $h_0(B/c_B) = qh_0(A/c_A)$ ,  $h_0(\bar{B}/c_B) = qh_0(\bar{A}/c_A)$  and  $h_0(\bar{B}) \leq qh_0(\bar{A})$ . It follows that  $v(B) - 1 \geq q(v(A) - 1)$  and we are done.

The following example shows that the case  $v(B) > v(A)$  can occur.

**Example 2.2.** Let  $C = \mathbb{C}[X, Y]/(Y^2 - X^3 + X)$  be the coordinate ring of the non singular cubic curve of  $\mathbb{A}_{\mathbb{C}}^2$ , let  $\mathfrak{m}$  be a maximal ideal of  $C$ . Put  $A' = C \oplus C$ ;  $\mathfrak{m}_1 = \mathfrak{m} \oplus C$  and  $\mathfrak{m}_2 = C \oplus \mathfrak{m}$  are maximal ideals of  $A'$ . Let  $A$  be the ring obtained from  $A'$  by glueing  $\mathfrak{m}_1$  with  $\mathfrak{m}_2$  (see [9]).  $A$  is a connected ring,  $A'$  is the integral closure of  $A$ ,  $A$  has no incidence cycles. There exists an étale covering  $h: C \rightarrow E$  of degree 2, with  $E$  a connected ring (see [10]). Let  $\mathfrak{M}'$  and  $\mathfrak{M}''$  be the two maximal ideals of  $E$  lying over  $\mathfrak{m}$ . Put  $B' = E \oplus E$ , let  $B$  be the ring obtained from  $B'$  by glueing  $\mathfrak{M}' \oplus E$  with  $E \oplus \mathfrak{M}'$  and  $\mathfrak{M}'' \oplus E$  with  $E \oplus \mathfrak{M}''$ . The étale covering  $h$  induces an étale covering  $f: A \rightarrow B$  of degree 2 (see [14]).  $B$  is a connected ring and it has an incidence cycle. We have  $v(A) = 0$  and  $v(B) = 1$ . Notice that  $A$  is QN, whereas  $B$  is SN but not QN.

The above example shows that quasinormality is not preserved by étale extensions, not even for finitely generated  $k$ -algebras, with  $k$  algebraically closed.

If the irreducible components of the curve  $X = \text{Spec}(A)$  are rational curves and if  $X$  has no incidence cycles, then any étale covering of  $X$  has no incidence cycles, but in this case it is a trivial covering.

**Proposition 2.3.** *Suppose that all irreducible components of  $\text{Spec}(A)$  are rational curves. Let  $f: A \rightarrow B$  be an étale covering, with  $B$  a connected ring. Let  $q$  be the degree of  $f$ . Then  $v(B) - 1 = q(v(A) - 1)$ .*

**Proof.** We have  $h_0(\bar{B}) = qh_0(\bar{A})$ . In fact let  $\mathfrak{P}$  be a minimal prime ideal of  $\bar{A}$ , we have  $q = \dim_{\bar{A}_{\mathfrak{P}}} \bar{B} \otimes_{\bar{A}} \bar{A}_{\mathfrak{P}}$ .  $\bar{A}_{\mathfrak{P}}$  is the residue field of  $\bar{A}$  at  $\mathfrak{P}$ , it is isomorphic to the quotient field of  $\bar{A}/\mathfrak{P}$ , which is  $k(T)$  by assumption. We have  $\bar{B} \otimes_{\bar{A}} \bar{A}_{\mathfrak{P}} \cong \bigoplus_{j=1}^r \bar{B}_{\mathfrak{P}_j}$ , where  $\{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$  is the set of the minimal prime ideals of  $\bar{B}$  lying over  $\mathfrak{P}$ . Since  $f$  is an étale homomorphism,  $\bar{B}_{\mathfrak{P}_j}$  is a separable finite extension of  $\bar{A} \cong k(T)$ . Hence we have  $\bar{B}_{\mathfrak{P}_j} \cong k(T)$  for all  $i = 1, \dots, r$ , because  $k(T)$  has not a proper separable finite extension (see [7]). It follows that  $r = q$  and then  $h_0(\bar{B}) = qh_0(\bar{A})$ . By using the same methods of the proof of Proposition 2.1, the conclusion follows.

**Corollary 2.4.** *Suppose that all irreducible components of  $\text{Spec}(A)$  are rational curves. Let  $f: A \rightarrow B$  be an étale covering. If  $v(A) = 0$ , then  $v(B) = 0$  and  $B$  is a trivial covering.*

**Proof.** We may suppose that  $A$  and  $B$  are connected. Let  $q$  be the degree of  $f$ . We have  $v(B) - 1 = q(v(A) - 1)$ ;  $v(A) = 0$  implies  $v(B) = 1 - q$ , which is non-negative, then  $q = 1$ .

In Proposition 2.3 and in Corollary 2.4, the assumption that the ground field  $k$  is algebraically closed cannot be omitted, as the following example shows.

Put  $A = \mathbb{R}[T^2 + 1, T(T^2 + 1)]$ .  $B = \mathbb{C}[T^2 + 1, T(T^2 + 1)]$  is a finite étale  $a$ -algebra. We have  $\bar{A} = \mathbb{R}[T]$ , the conductor  $c_A$  is a maximal ideal in  $A$  and in  $\bar{A}$ , hence  $v(A) = 0$ . We have  $\bar{B} = \mathbb{C}[T]$  and the conductor  $c_B$  is a maximal ideal in  $B$ , but it is the product of two maximal ideals in  $\bar{B}$ , hence  $v(B) = 1$ .

**Corollary 2.5.** *Let  $f: A \rightarrow B$  be an étale covering.*

- (a) *If  $B$  is QN, then  $A$  is QN.*
- (b) *If the irreducible components of  $\text{Spec}(A)$  are rational curves, then  $A$  is QN if and only if  $B$  is QN. In this case  $B$  is a trivial covering.*

**Proof.** The homomorphism  $f$  is faithfully flat. Then  $A$  is SN if and only if  $B$  is SN. The conclusion follows from Proposition 2.1 and Corollary 2.4.

## References

- [1] M. Artin, On the join of Hensel rings, *Advances in Math.* 7 (1971) 282–296.
- [2] H. Bass and P. Murty, Grothendieck groups and Picard groups of abelian group rings, *Ann. of Math.* 86 (1967) 16–73.
- [3] C. Berge, *Théorie des Graphes et ses Applications* (Dunod, Paris, 1958).
- [4] E. Davis, On the geometric interpretation of seminormality, *Proc. AMS* 68 (1978) 1–5.
- [5] S. Greco, Seminormality and quasinormality of group rings, *J. Pure Appl. Algebra* 18 (1980) 129–142.
- [6] S. Greco and C. Traverso, On seminormal schemes, *Compositio Math.* 40 (1980) 325–365.
- [7] A. Grothendieck, *Revetements Etale et Groupe Fondamental (SGA I)*, *Lecture Notes in Math.* 224 (Springer, Berlin–New York, 1971).
- [8] A. Grothendieck and J. Dieudonné, *Elements de Géométrie Algébrique, IV* (Publ. Math. Paris 1964/67).
- [9] C. Pedrini, Incollamenti di ideali primi e gruppi di Picard, *Rend. Sem. Mat. Univ. Padova* 48 (1973) 39–66.
- [10] M. Raynaud, *Annaux Locaux Henseliens*, *Lectures Notes in Math.* 169 (Springer, Berlin–New York, 1970).
- [11] L. Roberts, The  $K$ -theory of some reducible affine varieties, *J. Algebra* 35 (1975) 516–527.
- [12] L. Roberts, The  $K$ -theory of some reducible affine curves: A combinatorial approach, in: *Algebraic K-Theory*, *Lectures Notes in Math.* 551 (Springer, Berlin–New York, 1976).
- [13] C. Traverso, Seminormality and Picard group, *Ann. Sc. Norm. Sup. Pisa* 24 (1970) 585–595.
- [14] A. Vistoli, Incollamento di punti chiusi e gruppo fondamentale algebrico e topologico, *Rend. Sem. Mat. Univ. Padova*, to appear.